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The solitons in some geometrical field theories

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Abstract. Using the methods of differential geometry it is shown that the Born–Infeld scalar field in two-dimensional space-time and the relativistic string in three dimensions are described by the same non-linear Liouville equation $u_{tt} - u_{xx} = R\,\mathrm{e}^{u}$. This equation admits soliton solutions which may be stable or unstable, and there are periodical solutions among the stable ones. In the quantum case the solitons can be interpreted as massive particles, either stable or unstable with respect to the stability of the corresponding classical solution. The periodical soliton generates a series of resonances which have the equidistant mass spectrum. This result appears to be well suited to the theory of the closed relativistic string. In four dimensions the relativistic string is described by the same Liouville equation, but for the complex-valued function u.

1. Introduction

In recent years it has been discovered that a number of nonlinear fields admit soliton solutions (see e.g. Scott *et al* 1973, Whitham 1974). In elementary particle physics these solutions can be interpreted as particles which are different from the quanta of the initial field (Skyrme 1961, Dashen *et al* 1974a, b, 1975, Rajaraman 1975, Jackiw 1977); so one nonlinear field describes particles of several kinds. However, this can be demonstrated in the complete form only in nonlinear models, which are very far from physical reality. A classic example here is the field satisfying the sine–Gordon equation in two-dimensional space-time (Faddeev and Takhtajan 1974). The investigation of the soliton solutions even in such abstract models is of interest from a methodological viewpoint at least.

By means of differential geometry methods in this paper it is shown that the Born–Infeld massless scalar field in one space and one time dimension and the relativistic string moving in three-dimensional space-time are described by the nonlinear Liouville equation

$$u_{tt} - u_{xx} = R\,\mathrm{e}^{u}. \tag{1}$$

In differential geometry equation (1) is the Gauss equation (Stoker 1969) which connects the Gauss curvature of the surface $K = R/2$ with the coefficients of the first fundamental form of this surface

$$\mathrm{d}s^2 = \mathrm{e}^{u}(\mathrm{d}t^2 - \mathrm{d}x^2). \tag{2}$$

The Liouville equation (1) admits soliton-like solutions which may be stable or unstable, and there are periodical solutions among the stable ones. The addition of the term dependent on the soliton velocity to the canonical energy–momentum tensor allows us to define the total energy, momentum and rest mass of the soliton. In quantum theory these soliton solutions can be interpreted as massive particles, either stable or unstable with respect to the stability of the corresponding classical solution. The quanta of the initial field remain massless after the separation of the solitons. In the quantum case, the periodical soliton generates a series of resonances which have an equidistant mass spectrum beginning from the first excited state. This periodical soliton appears to be well suited to the theory of the closed relativistic string. The results obtained show that the nonlinear models containing only a massless field can generate a rich spectrum of massive particles and resonances, and this spectrum cannot be obtained in perturbation theory in principle.

2. The geometric approach to the Born–Infeld scalar field model and to the relativistic string theory

The Born–Infeld scalar field in one space and one time dimension has the action (Born and Infeld 1934)

$$S=-\gamma\int \mathrm{d}t\int \mathrm{d}x\,[1+\gamma^{-1}(\phi_x^2-\phi_t^2)]^{1/2}, \tag{3}$$

where $\phi_x=\partial\phi(x,t)/\partial x$, $\phi_t=\partial\phi(x,t)/\partial t$, and γ is a constant with the dimension of inverse length. The field $\phi(x,t)$ obeys the nonlinear equation

$$(\gamma-\phi_t^2)\phi_{xx}+2\phi_x\phi_t\phi_{xt}-(\gamma+\phi_x^2)\phi_{tt}=0, \tag{4}$$

which admits the wave solution of an arbitrary form propagating with the speed of light,

$$\phi(x,t)=\Phi(x\pm t).$$

The function Φ is not fixed by equation (4). By these properties equation (4) differs from the well investigated ones: the sine–Gordon equation, the Korteweg–de Vries equation and the nonlinear Schrödinger equation. All these equations have soliton solutions of an exactly fixed form and which propagate with an arbitrary velocity v (Scott *et al* 1973, Whitham 1974).

Now we present the so-called geometric approach to the Born-Infeld scalar field model. In this approach the model under consideration is described by the nonlinear Liouville equation (1) which has soliton solutions of a definite form.

In the work of Barbashov and Chernikov (1966a, b) the Born–Infeld scalar field was examined in parametrical representation by introducing the Lorentz vector $x_\mu(\sigma,\tau)$, with components

$$x^\mu(\sigma,\tau)=(t(\sigma,\tau),\,x(\sigma,\tau),\,y(\sigma,\tau)=\gamma^{-1/2}\phi(t(\sigma,\tau),x(\sigma,\tau))).$$

In the new variables, action (3) coincides with that for the infinite relativistic string in three-dimensional space-time (see e.g. Rebbi 1974, Scherk 1975),

$$S=-\gamma\iint \mathrm{d}\tau\,\mathrm{d}\sigma\,[(\dot{x}x')^2-\dot{x}^2x'^2]^{1/2}=-\gamma\iint\mathrm{d}\tau\,\mathrm{d}\sigma\,\sqrt{-g}, \tag{5}$$

where $\dot{x}_\mu=\partial x_\mu(\sigma,\tau)/\partial\tau$, $x'_\mu=\partial x_\mu(\sigma,\tau)/\partial\sigma$, $g=\det|g_{ij}|$, $g_{ij}=(\partial x_\mu/\partial u^i)\partial x_\mu/\partial u^j$ is the metric tensor on the string world surface $x_\mu(\sigma,\tau)$, $i,j=1,2$, $u^1=\tau$, $u^2=\sigma$, $\mu=1,2,3$.

The principle of least action, as applied to the functional (5), leads to the problem of determining the extremal surface in three-dimensional pseudo-Euclidean space

(t, x, y). On the surface examined the isothermal coordinate system can always be chosen in the form

$$\dot{x}^2 = g_{11} = -g_{22} = -x'^2, \qquad g_{12} = \dot{x}x' = 0. \tag{6}$$

In this case the equations of motion $\delta\sqrt{-g}/\delta x_\mu = 0$ are reduced to the D'Alambert equation for $x_\mu(\sigma, \tau)$,

$$\ddot{x}_\mu - x''_\mu = 0. \tag{7}$$

Instead of searching for the vector $x_\mu(\sigma, \tau)$ which describes the coordinates of the string world surface through the joint solution of the equations of motion (7) and of the nonlinear conditions (6), one can look for the first and second fundamental forms of the string world surface (Lund and Regge 1976, Lund 1977a, b, Omnes 1979, Barbashov and Koshkarov 1979). According to the basic theorem in the differential geometry of surfaces (Stoker 1969), the coefficients of these forms determine the position of the surface in the space up to translations and rotations. The first fundamental form is the squared interval between the two neighbouring points on the surface, and by virtue of (6) has the form

$$ds^2 = dx_\mu\, dx^\mu = g_{11}(d\tau^2 - d\sigma^2) = \dot{x}^2(d\tau^2 - d\sigma^2). \tag{8}$$

The second fundamental form is the length of the perpendicular from the given point of the surface to the tangent plane at the neighbouring point,

$$dl^2 = L\, d\tau^2 + 2M\, d\tau\, d\sigma + N\, d\sigma^2. \tag{9}$$

The functions L, M and N define the projections of the vectors $\ddot{\boldsymbol{x}}$, $\dot{\boldsymbol{x}}'$ and $\boldsymbol{x}''$ respectively onto the unit space-like vector $\boldsymbol{m}$ orthogonal to the $\dot{\boldsymbol{x}}$ and $\boldsymbol{x}'$,

$$\ddot{\boldsymbol{x}} = \Gamma^1_{11}\dot{\boldsymbol{x}} + \Gamma^2_{11}\boldsymbol{x}' + L\boldsymbol{m}, \qquad \dot{\boldsymbol{x}}' = \Gamma^1_{12}\dot{\boldsymbol{x}} + \Gamma^2_{12}\boldsymbol{x}' + M\boldsymbol{m}, \qquad \boldsymbol{x}'' = \Gamma^1_{22}\dot{\boldsymbol{x}} + \Gamma^2_{22}\boldsymbol{x}' + N\boldsymbol{m}, \tag{10}$$

where Γ^i_{jk} are the Christoffel symbols of the second kind of the string world surface,

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \tfrac{1}{2}g^{il}\Gamma_{l,jk} = \tfrac{1}{2}g^{il}(\partial g_{lj}/\partial u^k + \partial g_{lk}/\partial u^j - \partial g_{jk}/\partial u^l), \tag{11}$$

in which $i, j, k, l = 1, 2$, $u^1 = \tau$, $u^2 = \sigma$. From equations (7) and (10) it follows that

$$L = N. \tag{12}$$

The arbitrariness in the choice of the coordinate system on the surface examined which, remains after imposing conditions (6), can be used to fix the coefficients of the second fundamental form,

$$(\ddot{x} \pm \dot{x}')^2 = \psi''^2_\pm(\sigma \pm \tau) = -q^2_\pm, \tag{13}$$

where $\psi_\pm$ are arbitrary functions in the general solution of equation (7),

$$x_\mu(\sigma, \tau) = (\psi_{+\mu}(\sigma + \tau) + \psi_{-\mu}(\sigma - \tau))/2.$$

Taking into account the invariance of equations (6) and (7) under the conformal transformations $\tilde{\sigma} \pm \tilde{\tau} = f_\pm(\sigma + \tau)$, it can easily be shown that condition (13) may always be satisfied by the corresponding choice of the functions $f_\pm$ (Barbashov and Koshkarov 1979). The quantities $q_\pm$ are arbitrary functions of the variables $\sigma \pm \tau$ given beforehand. For simplicity we shall take these functions as constants.

Substituting expansions (10) into (13) and taking into account that $\boldsymbol{m}^2=-1$, we obtain

$$(L\pm M)^2=q_{\pm}^2. \tag{14}$$

So, only the coefficient $g_{11}=\dot{x}^2$ remains unfixed in the first and second fundamental forms. It must be defined from the Gauss and Codazzi equations connecting the coefficients of both fundamental forms of the surface.

The Codazzi equations in the case under consideration have the form

$$\partial L/\partial\sigma-\partial M/\partial\tau=0, \qquad \partial M/\partial\sigma-\partial L/\partial\tau=0;$$

hence it follows that

$$L=(\lambda_+(\sigma+\tau)+\lambda_-(\sigma-\tau))/2, \qquad M=(\lambda_+(\sigma+\tau)-\lambda_-(\sigma-\tau))/2, \tag{15}$$

where the $\lambda_\pm$ are arbitrary functions. It is obvious that (15) does not contradict (14) if we put $q_\pm=\lambda_\pm$.

Now only the Gauss equation remains. It can be obtained more easily in the following way. Using equations (6), (12) and (14) we find the Gauss curvature of the string world surface,

$$K=\frac{LN-M^2}{g_{11}g_{22}-g_{12}^2}=\frac{L^2-M^2}{-g_{11}^2}=\frac{q_+q_-}{(\dot{\boldsymbol{x}}^2)^2},$$

and substitute it into equation (1),

$$u_{\tau\tau}-u_{\sigma\sigma}=2(q_+q_-)\,\mathrm{e}^u, \tag{16}$$

where $\mathrm{e}^{-u}=g_{11}=\dot{\boldsymbol{x}}^2$.

From a comparison between (16) and (1) there follows a direct connection between the internal geometries of the minimal surfaces and of the surfaces with a constant Gauss curvature in a three-dimensional pseudo-Euclidean space E_3^1: the products of the corresponding coefficients of the first fundamental forms of these surfaces are equal to unity.

Equation (16) has to be complemented by the boundary conditions if the relativistic string is finite. For example, for a closed string, $0\leqslant\sigma\leqslant\pi$, we have

$$u(0,\tau)=u(\pi,\tau).$$

3. The relativistic string in four-dimensional space–time

Our consideration will be based on the following embedding theorem of differential geometry (Eisenhart 1949). If the symmetrical tensor g_{ij}, p symmetrical tensors $b_{\alpha|ij}$ and $p(p-1)/2$ vectors $\nu_{\alpha\beta|i}(=-\nu_{\beta\alpha|i})$, $i,j=1,2$, $\alpha,\beta=3,4,\ldots,p+2$, determine the two-dimensional surface V_2 with the fundamental tensor g_{ij}, embedded into the real flat space S_{p+2} (the Riemann curvature tensor of this space vanishes identically), then and only then are the following equations satisfied:

$$R_{ijkl}=\sum_\alpha e_\alpha(b_{\alpha|ik}b_{\alpha|jl}-b_{\alpha|il}b_{\alpha|jk}), \tag{17}$$

$$b_{\alpha|ij;k}-b_{\alpha|ik;j}=\sum_\beta e_\beta(\nu_{\beta\alpha|k}b_{\beta|ij}-\nu_{\beta\alpha|j}b_{\beta|ik}), \tag{18}$$

$$\nu_{\beta\alpha|j;k} - \nu_{\beta\alpha|k;j} + \sum_{\gamma} e_{\gamma}(\nu_{\gamma\beta|j}\nu_{\gamma\alpha|k} - \nu_{\gamma\beta|k}\nu_{\gamma\alpha|j}) + g^{lm}(b_{\beta|lj}b_{\alpha|mk} - b_{\beta|lk}b_{\alpha|mj}) = 0. \tag{19}$$

Here R_{ijkl} is the Riemann–Christoffel curvature tensor of the two-dimensional surface V_2 which has only one essential component,

$$R_{1212} = \tfrac{1}{2}(2\partial^2 g_{12}/\partial u^1 \partial u^2 - \partial^2 g_{11}/\partial u^2 \partial u^2 - \partial^2 g_{22}/\partial u^1 \partial u^1) + g^{lm}(\Gamma_{m,21}\Gamma_{l,12} - \Gamma_{m,22}\Gamma_{l,11}), \tag{20}$$

the $\Gamma_{i,jk}$ are the Christoffel symbols (see formula (11)), and the e_α are the constants equal to ± 1. The semicolon in equations (18) and (19) means covariant differentiation with respect to the metric tensor

$$g_{ij} = \sum_{\mu=1}^{p+2} C_\mu \frac{\partial x^\mu}{\partial u^i}\frac{\partial x^\mu}{\partial u^j}, \qquad g_{ik}g^{kj} = \delta^j_i.$$

The constants C_μ, $\mu = 1, 2, \ldots, p+2$ take into account the metric signature of space S_{p+2} and they equal ± 1. In the theory of the relativistic string, S_{p+2} is the Minkowsky space E^1_{p+2}, so we put $C_1 = -C_\nu = 1$, $\nu = 2, 3, \ldots, p+2$. At any point of the string world surface V_2, embedded into the flat space S_{p+2}, there can be constructed the set of p orthonormal unit vectors η^μ_α which are orthogonal to the tangent vectors $\dot{x}^\mu$ and x'^μ of the world surface of the string:

$$\sum_{\mu=1}^{p+2} C_\mu \eta^\mu_\alpha \eta^\mu_\beta = \begin{cases} e_\alpha, & \alpha = \beta \\ 0, & \alpha \neq \beta \end{cases},$$

$$\sum_{\mu=1}^{p+2} C_\mu \eta^\mu_\alpha \dot{x}^\mu = \sum_{\mu=1}^{p+2} C_\mu \eta^\mu_\alpha x'^\mu = 0, \qquad \alpha, \beta = 3, 4, \ldots, p+2.$$

In the chosen metric of space S_{p+2}, the constants e_α are equal to $+1$ for the time-like vectors η^μ_α and $e_\alpha = -1$ for the space-like vectors η^μ_α. From the physical viewpoint in the string theory (Rebbi 1974, Scherk 1975) one puts $g_{11} = \dot{x}^2 > 0$, $g_{22} = x^2 < 0$, and all vectors orthogonal to $\dot{x}$ and x' are space-like, and as a consequence $e_\alpha = -1$, $\alpha = 3, 4, \ldots, p+2$.

The world sheet of the string is the minimal surface, so in the isothermal coordinate system (6) the vector $x^\mu(\sigma, \tau)$ which describes this surface obeys the D'Alambert equation (7). If the surface is embedded into the three-dimensional flat space ($p = 1$), then there are no vectors $\nu_{\alpha\beta|i}$, and equations (17)–(19) are reduced to the first two, in this case (17) which is the Gauss equation and (18) which is the Codazzi equation.

In the case under consideration in coordinate system (6), equation (17), by virtue of (20), takes the form

$$\ddot{g}_{11} - g''_{11} - g^{11}(\dot{g}^2_{11} - g'^2_{11}) = -2\sum_{\alpha=3}^{p+2}(b_{\alpha|11}b_{\alpha|22} - b^2_{\alpha|12}). \tag{21}$$

The tensors $b_{\alpha|ij}$ are defined by the derivation formulae

$$x^\mu_{;ij} = \sum_{\alpha=3}^{p+2} e_\alpha b_{\alpha|ij}\eta^\mu_\alpha, \tag{22}$$

where η^μ_α is the set of p orthonormal unit vectors introduced above. Taking (7) into account we obtain from expansion (22)

$$b_{\alpha|11} = b_{\alpha|22}, \qquad \alpha = 3, 4, \ldots, p+2.$$

In addition to (6), one can impose on the variables $x^{\mu}(\sigma, \tau)$ the gauge conditions (13). Substituting (22) into (13) we obtain

$$\sum_{\alpha=3}^{p+2} (\beta_{\alpha|11} \pm b_{\alpha|12})^2 = q_{\pm}^2.$$

If the string moves in four-dimensional Minkowsky space ($p = 2$, $\alpha = 3, 4$), then the variables

$$b_{3|11} \pm b_{3|12} = q_{\pm} \cos \alpha_{\pm}, \qquad b_{4|11} \pm b_{4|12} = q_{\pm} \sin \alpha_{\pm}$$

can be introduced. Equations (21), (18) and (19) now take the form

$$\ddot{g}_{11} - g''_{11} - g^{11}(\dot{g}_{11}^2 - g'^2_{11}) = -2(q_+q_-)\cos(\alpha_+ - \alpha_-), \tag{23}$$

$$\dot{\alpha}_+ - \alpha'_+ = \nu_1 - \nu_2, \qquad \dot{\alpha}_- + \alpha'_- = \nu_1 + \nu_2, \tag{24}$$

$$\nu'_1 - \dot{\nu}_2 + g^{11}(q_+q_-)\sin(\alpha_+ - \alpha_-) = 0, \tag{25}$$

where $\nu_1 = \nu_{43|1}$, $\nu_2 = \nu_{43|2}$. In terms of variables $\theta = \alpha_+ - \alpha_-$ and $g_{11} = \mathrm{e}^{-u}$, equations (23) and (25) part from the system (23)–(25):

$$\ddot{u} - u'' = 2(q_+q_-)\,\mathrm{e}^{u}\cos\theta, \tag{26}$$

$$\ddot{\theta} - \theta'' = 2(q_+q_-)\,\mathrm{e}^{u}\sin\theta. \tag{27}$$

These equations can be reduced to one by using the complex-valued function $w = u + \mathrm{i}\theta$,

$$\ddot{w} - w'' = R\,\mathrm{e}^{w}, \tag{28}$$

where $R = 2(q_+q_-)$. So the theory of the relativistic string in four-dimensional space–time in gauge (13) is again reduced to the Liouville equation. But in contrast with the three-dimensional case (see formula (16)), this equation is for the complex-valued function w.

Let us pass to an arbitrary dimension ($p > 2$) of space–time into which the world sheet of the string is embedded. We shall consider the sets of variables $b_{\alpha|11}$ and $b_{\alpha|12}$, $\alpha = 3, 4, \ldots, p+2$ as coordinates of p-dimensional Euclidean vectors $b^1 = (b_{3|11}, b_{4|11}, \ldots, b_{p+2|11})$ and $b^2 = (b_{3|12}, b_{4|12}, \ldots, b_{p+2|12})$. Introducing the variable θ as the angle between these vectors, we can reduce (17) to (26) again. Let us show that the particular solution of the system (17)–(19) is $\theta = 0$, $\nu_{\beta\alpha|j} = 0$ and $b_{\alpha|ij}$ are the constants. Equation (18) is satisfied identically in this case, and (19) gives

$$b_{\alpha}^1/b_{\beta}^1 = b_{\alpha}^2/b_{\beta}^2, \qquad \alpha \neq \beta, \quad \alpha, \beta = 3, 4, \ldots, p+2.$$

This is also identical, as for $\theta = 0$ we have $b_{\alpha}^1 = \lambda b_{\alpha}^2$, where λ is constant. Now the essential equation is equation (26) which is reduced to (16). So in the case of space–time with any dimension there are such string motions which are described by one real Liouville equation (16).

4. Investigation of the soliton solutions

Now we shall study the fundamental equation (1) for the real function $u(x, t)$. The general solution of this equation, obtained by Liouville (1853), is

$$\mathrm{e}^{u(x,t)} = 8f'(x+t)g'(x-t)/R(f(x+t) - g(x-t))^2, \tag{29}$$

where f and g are arbitrary functions and the prime denotes differentiation with respect to the function argument. However, the general solution of equation (1) is not of great interest physically. From this point of view the particular solutions of (1)—the solitons†—are more attractive. They have the form

$$u(x, t) = F(x - vt), \tag{30}$$

where $|v| < 1$. It is not easy to see these solutions in formula (29). It is simpler to insert (30) into (1), which gives the following ordinary differential equation for F:

$$(1 - v^2)F'' = -R\, e^F.$$

If $R > 0$, then we obtain

$$e^{u_1} = (m^2/32R)\, \mathrm{sech}^2[(mx - vt - x_0)/8\sqrt{1 - v^2}], \tag{31}$$

where m is an arbitrary constant. As will be shown below, m is the mass of the soliton (31). If $R < 0$, then we obtain two kinds of soliton:

$$e^{u_2} = (m^2/32|R|)\, \mathrm{cosech}^2[(mx - vt - x_0)/8\sqrt{1 - v^2}], \tag{32}$$

$$e^{u_3} = (m^2/32|R|)\, \mathrm{cosec}^2[(mx - vt - x_0)/8\sqrt{1 - v^2}]. \tag{33}$$

With the obvious solutions (31)–(33), one can easily choose the functions f and g in the general solution (29) so that (29) will result in (31)–(33). For example, for soliton (31), taking into account the formula

$$\mathrm{sech}^2\, \tfrac{1}{2}(y - z) = 4\, e^y\, e^z/(e^y + e^z)^2,$$

we have to put

$$f(x + t) = (8/m) \exp[m\sqrt{(1 - v)/(1 + v)}(x + t - x_0)/8],$$

$$g(x - t) = -(8/m) \exp[-m\sqrt{(1 + v)/(1 - v)}(x - t - x_0)/8].$$

The solutions (31) and (32) are solitary waves, moving with a velocity less than the speed of light. Solution (33) is the periodical soliton, describing the 'comb' of the waves. It should be noted that (32) and (33) have the non-integrable singularities

$$e^{u_i} \sim z^{-2}, \qquad \text{when } z \to 0, \quad z = (x - vt - x_0)/\sqrt{1 - v^2}, \quad i = 1, 2,$$

and thereby they do not formally satisfy the requirements usually imposed on soliton solutions (Scott *et al* 1973, Whitham 1974). However, to simplify the terminology, we shall call these solutions 'solitons', and it will be shown that their singularities do not lead to the principal difficulties in attempting to interpret these solutions as extended particles.

Apart from one-soliton solutions, equation (1) also has n-soliton solutions (Andreev 1976). Such a solution describes one soliton moving with arbitrary velocity and $(n - 1)$ solitons moving with the speed of light. Only soliton (31) follows from the n-soliton solution at $n = 1$, while the other solitons (32), (33) cannot be obtained in such a way. For simplicity, we shall restrict our consideration to the one-soliton solutions only.

The Liouville equation (1) is the Euler equation for the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(u_t^2 - u_x^2) + R\, e^u. \tag{34}$$

† Equation (2) in variables $x \pm t$ has been studied by the inverse scattering method (Andreev 1976). However, the soliton solutions of this equation have not been discussed from the viewpoint of particle physics.

The potential energy in this Lagrangian, $V(u)=-R\,\mathrm{e}^{u}$, is a monotonic function of the variable u. Usually, the relativistic invariant soliton-like solutions are considered in the models with spontaneous symmetry-breaking, where $V(u)$ has at least two minima (Dashen *et al* 1974b, Rajaraman 1975, Jackiw 1977). This difference between the model under consideration and the ones usually studied leads to some peculiarities. For instance, it is impossible to introduce the topological charge for the solitons of equation (1) by standard methods.

Now let us show that by using the Lagrangian (34) we can define the total energy, momentum and mass of the solutions (31)–(33) in a correct relativistic correlation. We take the energy–momentum tensor for the solitons in the form

$$\Theta^{\mu\nu}=T^{\mu\nu}+(m^2/32)\eta^{\mu\nu}-(m^2/16)v^{\mu}v^{\nu}, \tag{35}$$

where $T^{\mu\nu}$ is the canonical energy–momentum tensor of the field $u(x,t)$, corresponding to the Lagrangian density (34):

$$T^{\mu\nu}=(\partial\mathscr{L}/\partial(\partial_\mu u))\partial^\nu u-\eta^{\mu\nu}\mathscr{L},$$

where $\mu,\nu=0,1$, $\eta^{00}=-\eta^{11}=1$, v^{μ} is the velocity vector of the soliton, $v^0=1/\sqrt{1-v^2}$ and $v^1=v/\sqrt{1-v^2}$. The term added to $T^{\mu\nu}$ in formula (35) does not depend on the coordinates x, t and leads to convergent integrals at infinity for the energy and momentum of the solitons:

$$p^i=\int\Theta^{i0}[u_c(x,t)]\,\mathrm{d}x. \tag{36}$$

Substituting the solutions $u_i(x,t)$ of (31)–(33) into (36) we obtain

$$\begin{aligned} &E_1=-m/\sqrt{1-v^2}, \qquad P_1=-mv/\sqrt{1-v^2},\\ &E_i=mI_i/\sqrt{1-v^2}, \qquad P_i=mI_iv/\sqrt{1-v^2}, \qquad i=2,3, \end{aligned} \tag{37}$$

where $I_2=\int_0^\infty \mathrm{d}x\,\mathrm{cosech}^2\,x$, $I_3=\int_0^{\pi/2}\mathrm{d}x(\cot^2 x-1)$. So in the first case one can consider the constant m as the mass of the soliton. For the solitons (32) and (33) the masses are equal to mI_i, $i=2,3$. The integrals I_i, divergent at zero, can be regularised by introducing a cut-off, for example. However, it will be sufficient that between E_i and P_i in formulae (37) a correct relativistic correlation does exist. Just this fact enables us to interpret the solitons as extended particles with non-zero rest mass even in the classical theory.

Let us turn to the investigation of the soliton stability. At first, we consider the static soliton solutions which are defined by formulae (31)–(33) at $v=0$. We represent the solution of equation (1) in the form

$$u(x,t)=u_i(x)+\mathrm{e}^{\mathrm{i}\omega t}\psi(x). \tag{38}$$

Substituting (38) into (1) leads to an equation for $\psi(x)$ which has the form of the one-dimensional Schrödinger equation with the potential $V[u_i(x)]=-R\,\mathrm{e}^{u_i(x)}$:

$$(-\mathrm{d}^2/\mathrm{d}x^2+V[u_i(x)]\psi(x)=\omega^2\psi(x), \tag{39}$$

where

$$V[u_1(x)]=-(m^2/32)\,\mathrm{sech}^2(mx/8), \tag{40}$$

$$V[u_2(x)]=(m^2/32)\,\mathrm{cosech}^2(mx/8), \tag{41}$$

$$V[u_3(x)]=(m^2/32)\,\mathrm{cosec}^2(mx/8). \tag{42}$$

If in equation (39) $\omega^2 > 0$, the solution $u_i(x)$ is stable in the classical theory, and by virtue of the relativistic invariance $u_i(x, t)$ is stable as well. When $\omega^2 < 0$, the correction to $u_i(x)$ in (38) increases exponentially with time, and the soliton is unstable.

Equation (39) with potentials (40)–(42) can be solved exactly (Flügge 1971, Morse and Feshbach 1953). In the first case ω^2 has one negative value, $\omega_1^2 = -m^2/64$, and a continuous spectrum beginning from the translation mode $\omega_0^2 = 0$. Because $\omega_1^2 < 0$, the soliton solution $u_1(x, t)$ is already unstable in classical theory.

For potential (41) ω^2 has the continuous spectrum $\omega^2 > 0$ and the translation mode $\omega_0^2 = 0$ which again adjoins the continuous spectrum. The solution $u_2(x, t)$ is stable.

Potential (42) can be reduced to the Pöschl–Teller potential (Flügge 1971). The periodicity of this potential turns out to be unessential for the solutions of equation (39), as the neighbouring potential wells are separated by impenetrable barriers. Therefore we can restrict ourselves to a consideration of one of these wells. In this case there are only the discrete spectrum $\omega_n^2 = m^2(n+1)^2/64$, $n = 1, 2, \ldots$ and the zero-frequency mode $\omega_0 = 0$. The solution $u_3(x, t)$ is stable.

So in classical theory the soliton $u_1(x, t)$ can be considered as an unstable particle with mass m and a lifetime of about $8m^{-1}$. The solitons $u_2(x, t)$ and $u_3(x, t)$ describe stable particles.

Let us go to the discussion of equation (28) for the complex function w. Obviously, the Liouville solution (29) can be generalised to this equation if we consider the functions f and g to be complex-valued,

$$f(x+t) = f_1(x+t) + \mathrm{i} f_2(x+t), \qquad g(x-t) = g_1(x-t) + \mathrm{i} g_2(x-t).$$

In this case the solution of the system (26), (27) is

$$\mathrm{e}^{u(x,t)/2} = \frac{64}{R^2} \frac{(f_1'^2 + f_2'^2)(g_1'^2 + g_2'^2)}{[(f_1-g_1)^2 + (f_2-g_2)^2]^2},$$

$$\theta(x, t) = \tan^{-1}\frac{f_2'}{f_1'} + \tan^{-1}\frac{g_2'}{g_1'} - \tan^{-1}\frac{2(f_1-g_1)(f_2-g_2)}{(f_1-g_1)^2 - (f_2-g_2)^2}.$$

Equation (28), as in the case of the real function w, has the soliton solution

$$\mathrm{e}^{w/2} = A\sqrt{\frac{2}{R}}\,\mathrm{sech}\left(A\frac{\sigma - v\tau}{\sqrt{1-v^2}} + \delta\right), \tag{43}$$

where A and δ are arbitrary complex constants: $A = a_1 + \mathrm{i}a_2$, $\delta = \delta_1 + \mathrm{i}\delta_2$. Separating the real and imaginary parts in (43), we obtain

$$\mathrm{e}^u = \frac{4(a_1^2 + a_2^2)}{|R|(\cosh 2z_1 + \cos 2z_2)}, \qquad \tan\tfrac{1}{2}\theta = \frac{a_2 + a_1 \tanh z_1 \tan z_2}{a_1 - a_2 \tanh z_1 \tan z_2},$$

where $z_i = a_i(\sigma - v\tau)/\sqrt{1-v^2} + \delta_i$, $i = 1, 2$.

Equations (26) and (27) are the Euler equations for the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\dot{u}^2 - u'^2) - \tfrac{1}{2}(\dot{\theta}^2 - \theta'^2) + R\,\mathrm{e}^u \cos\theta. \tag{44}$$

The corresponding Hamiltonian is

$$\mathscr{H} = \tfrac{1}{2}(\pi_u^2 - u'^2) - \tfrac{1}{2}(\pi_\theta^2 + \theta'^2) + R\,\mathrm{e}^u \cos\theta,$$

where $\pi_u = \partial\mathscr{L}/\partial\dot{u} = \dot{u}$ and $\pi_\theta = \partial\mathscr{L}/\partial\dot{\theta} = -\dot{\theta}$. The free Hamiltonian of the field θ is included into $\mathscr{H}$ with sign minus.

5. Quantum theory

There are many approaches to the construction of quantum theory for field models with particle-like classical solutions (see e.g. Dashen *et al* 1974a, b, 1975, Faddeev and Korepin 1975, Goldstone and Jackiw 1975, Rajaraman 1975). The ideas of these methods differ at first sight, but the basic equations determining the spectrum of the states turn out to be practically the same. Without going into details, we shall follow the so-called canonical quantisation of the particle-like solutions, which is closer to the usual field theory approach, although somewhat formal (Tomboulis 1975). In this section only equation (1) will be considered for the real function u.

The field $u(x, t)$ will be represented in the form

$$u(x, t) = u_c(x - x_0) + \psi(x, t), \tag{45}$$

where $u_c(x - x_0)$ are the soliton solutions (31)–(33) with $v = 0$. We shall consider as coordinates of the system the centre-of-mass position of the soliton $x_0(t) = x_0 + vt$ and the field $\psi(x, t)$. The canonical conjugate momenta are $p(t)$ and $\pi(t)$, respectively.

After substituting (45) into (35), the total Hamiltonian can be divided into the free term and the interaction Hamiltonian,

$$H = H_0 + H_\mathrm{I},$$

where $H_0 = m + \frac{1}{2}\int \mathrm{d}x\, (\pi^2 + \psi'^2 - \psi^2 R\, \mathrm{e}^{u_c})$ is the Hamiltonian of the particle with mass m and of the field $\psi(x, t)$, imbedded into the external classical field $-R\,\mathrm{e}^{u_c}$. The Hamiltonian H_I, which describes the interaction of $\psi(x, t)$ with the soliton, depends on x_0, p, ψ and π. The explicit form of H_I is complicated (Tomboulis 1975), and we do not write it here.

It follows from the Hamiltonian equations with H_0 that

$$\ddot{\psi} - \psi'' = R\, \mathrm{e}^{u_c}\psi.$$

Below we shall use the usual method of quantisation in an external field (Schweber 1961) by the expansions

$$\psi(x, t) = \sum_k \frac{1}{\sqrt{2\omega_k}} (b_k \psi_k(x)\, \mathrm{e}^{-\mathrm{i}\omega_k t} + b_k^+ \psi_k^+(x)\, \mathrm{e}^{\mathrm{i}\omega_k t}),$$

$$\pi(x, t) = \sum_k (-i)\sqrt{\tfrac{1}{2}\omega_k} (b_k \psi_k(x)\, \mathrm{e}^{-\mathrm{i}\omega_k t} - b_k^+ \psi_k^+(x)\, \mathrm{e}^{\mathrm{i}\omega_k t}),$$

where the $\psi_k(x)$ compose a complete set of solutions of equation (39) with excluded translation mode:

$$\sum_k \psi_k(x)\psi_k^+(y) = \delta(x - y) - u_c'(x)u_c'(y)/m.$$

The symbol Σ_k also represents here the integration over k if necessary. The canonical commutation relations

$$[b_k, b_{k'}^+] = \delta_{kk'}, \qquad [b_k, b_{k'}] = [b_k^+, b_{k'}^+] = 0$$

are postulated, and the Hilbert space of the states $|P, \{k_i\}\rangle = b^+_{\{k_i\}}|P\rangle$ is constructed. Here P is a common momentum and $\{k_i\}$ is a set of meson momenta. The perturbation theory can be developed on this basis as usual.

After the transition to the normal product of the operators b_k, the free Hamiltonian H_0 takes the form

$$H_0 = m + \sum_k \omega_k b^+_k b_k. \tag{46}$$

Calculating the matrix elements of H_0 over the state vectors $|P, \{k_i\}\rangle$, we obtain the energy spectrum of the system in the zero approximation of the perturbation theory. It is obvious that this spectrum is defined completely by the eigenvalues in equation (39).

The soliton (31), being unstable in classical theory, will be unstable in the quantum case too, because the correction to the energy (46) from the continuous spectrum is purely imaginary in this case.

In quantum theory the soliton (32) corresponds to the stable particle with mass m, and the field $\psi(x, t)$ describes the massless mesons. In the model under consideration there is no conservation law of the soliton number (the topological charge), so the quantum transitions of the soliton particles into the meson states are not in principle forbidden.

The periodical soliton (33) gives the richest spectrum,

$$E_0 = m, \qquad E_n = m + m(n+1)/8, \qquad n = 1, 2, \ldots.$$

It is interesting that the spectrum is equidistant if we do not consider the lowest state with energy E_0. The spacing of the energy levels is defined by the soliton mass $m/8$ and can take any value. The states of this spectrum are time-independent only if the interaction H_{I} is neglected. The Hamiltonian H_{I} leads to the transition between these states, and in reality we have a series of resonances.

The periodical soliton and the spectrum created by it are well suited to the theory of the closed relativistic string (Rebbi 1974, Scherk 1975). In the usual approach, this model has the equidistant spectrum of the stationary states which form the basis for the construction of the dual resonance models. The zero width of the energy levels is an essential defect of these models. In this connection, the mass spectrum obtained by taking into account the soliton solutions in the theory of the relativistic string is more realistic.

6. Conclusions

The basis of the geometrical approach to the relativistic string theory and to the Born–Infeld scalar field model is the change of the variables $x_\mu(\sigma, \tau)$ to one function $u(\sigma, \tau) = -\ln \dot{x}^2$. Mathematically, the string coordinates $x_\mu(\sigma, \tau)$ and the function $u(\sigma, \tau)$ carry the same information about the dynamics of the system because we can reconstruct $x_\mu(\sigma, \tau)$ from the known function $u(\sigma, \tau)$ by integrating formulae (10) or (22). However, this transition cannot be considered, as the canonical transformation and quantisation in terms of the variables $x_\mu(\sigma, \tau)$ and and $u(\sigma, \tau)$ gives different results. In this connection, a question arises: what are the variables which should be used to quantise the nonlinear models? The physical evaluation of the final results will probably be the only decisive criterion here.

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